Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 807459, 11 pages
http://dx.doi.org/10.1155/2013/807459

Research Article

Regularity of a Stochastic Fractional Delayed Reaction-Diffusion Equation Driven by Lévy Noise

Tianlong Shen, Jianhua Huang, and Jin Li

Department of Mathematics, National University of Defense Technology, Changsha 410073, China

Correspondence should be addressed to Jianhua Huang; jhhuang32@nudt.edu.cn

Received 27 June 2013; Accepted 18 August 2013

Academic Editor: Agacik Zafer

The current paper is devoted to the regularity of the mild solution for a stochastic fractional delayed reaction-diffusion equation driven by Lévy space-time white noise. By the Banach fixed point theorem, the existence and uniqueness of the mild solution are proved in the proper working function space which is affected by the delays. Furthermore, the time regularity and space regularity of the mild solution are established respectively. The main results show that both time regularity and space regularity of the mild solution depend on the regularity of initial value and the order of fractional operator. In particular, the time regularity is affected by the regularity of initial value with delays.

1. Introduction

Recently, fractional partial differential equations attract more and more attention. They appear more and more frequently in different research areas and engineering applications. They have been applied to model various phenomena in image analysis, risk management, and statistical mechanics (see, e.g., [1, 2]). There are many papers concerning the existence and regularity of the solution for fractional Navier-Stokes, fractional Ginzburg-Landau equation, fractional Burgers equation, fractional Langevin equation, and so on (see [3, 4] and references therein).

Stochastic partial differential equations driven by Gaussian noise and non-Gaussian noise such as Lévy noise have also attracted a lot of attention. It seems more significant to investigate fractional partial differential equations with some random force, and some authors have investigated the existence and regularity of the solutions for stochastic fractional partial differential equations ([2, 5–7] and the references therein). The authors in [6, 7] proved the existence and uniqueness of the solution for a stochastic fractional partial differential equation driven by a space-time white noise in one dimension. Truman and Wu in [8] applied the Banach fixed point theorem to show the existence and uniqueness of the mild solution for fractal Burgers equations driven by Lévy noise on real line. Brzeźniak and Debbi in papers [9, 10] proved the existence and ergodicity of the solution for fractal Burgers equation driven by Gaussian space-time white noise, and we refer to [9, 10] for more details. In mathematical biology and other fields, delays are often considered in the model such as maturation time for population dynamics. Some efforts have been devoted to the development of the theory of PDEs with delay. Such equations are naturally more difficult since they are infinite dimensional both in time and space variables. We refer to the monographs [11, 12] for more details. To our knowledge, there is no paper to study the stochastic fractional reaction-diffusion equation with delays.

It is worth to point out that the authors in [8] study the existence of the mild solution for stochastic fractional Burgers equation driven by Lévy noise, but they could not provide the regularity of the mild solution. The authors in [7, 13] study the regularity of the mild solution for stochastic fractional partial differential equations driven by Gaussian white noise, but not Lévy noise. There is a natural question, how about the regularity of the mild solution for the stochastic fraction delayed reaction-diffusion equation driven by Lévy noise?

Motivated by [8], in the present paper, we will study the stochastic fractional reaction-diffusion equation with delays driven by Lévy process followed as:

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \lambda \Delta_{\alpha} u(t,x) + f(t,x,u_t) \\ &\quad + g(t,x,u_t) Z_{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), &\qquad u(\eta,x) = \phi(\eta,x), \quad \eta \in [-r,0], \end{aligned} \tag{1}$$

where $\Delta_\alpha := -(-d^2/dx^2)^{\alpha/2}$ is the fractional Laplacian operator with $\alpha \in (0,2]$, the constants $\lambda \in \mathbb{R}$, $f, g : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable, the function $u_t = u(t+\eta)$, and $Z_{t,x}$ is the one-dimensional Lévy process (see Section 2 for the definition). Recall that D_α reduces to be the Laplacian operator when $\alpha = 2$.

In this paper, the existence, uniqueness, time regularity, and space regularity of the mild solution for (1) are shown for $\alpha \in (1,2]$ in the proper working function space which is affected by the delays. The main results show that both time regularity and space regularity of the mild solution for (1) depend on the regularity of initial value and the order of fractional operator. In particular, the time regularity is affected by the the regularity of initial value with delays.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of the Lévy space-time white noise. Then, some useful properties for the fractional Green kernel are presented. In Section 3, the proper working function space is constructed. Then the existence and uniqueness of the mild solution for (1) are proved by the Banach fixed point theorem in the proper working function space. Finally, the time regularity and space regularity of the mild solution are provided, respectively, in Section 4.

2. Preliminaries

In this section, we first introduce the Lévy space-time white noise. Then, some useful properties for the fractional Green kernel are presented.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual condition. For one-dimensional Lévy process $Z_{t,x}$, it follows from Lévy-Itô decomposition that there exist a constant β_1 and a nonnegative constant β_2, and a one-dimensional space-time white noise $W_{t,x} = (\partial^2 W/\partial t \partial x)(t,x)$ ($W(t,x)$ is a Brownian sheet on $[0,\infty) \times \mathbb{R}$) such that

$$Z_{t,x} = \beta_1 t + \beta_2 W_{t,x} + \int_{|z|<1} z\widetilde{N}(t,x,dz) + \int_{|z|\geq 1} zN(t,x,dz), \tag{2}$$

where

$$\widetilde{N}(t,x,A) := N(t,x,A) - t\nu(A), \tag{3}$$

where $\nu(A) := E[N(1,A)]$ is the Lévy measure of $Z_{t,x}$.

Similar to [14], for any p, we denote

$$\widehat{c}_p := \left(\int_{\mathbb{R}} |z|^p \nu(dz)\right)^{1/p}. \tag{4}$$

In what follows, we assume that

$$\widehat{c} := \sup_{p\geq 1} \widehat{c}_p < \infty. \tag{5}$$

Recalling that

$$\int_{|z|<1} zN(t,x,dz) = \int_{|z|<1} z\widetilde{N}(t,x,dz) + t\int_{|z|<1} z\nu(dz). \tag{6}$$

By absorbing $\widetilde{\beta} := -\int_{|z|<1} z\nu(dz)$ into β_1, we can rewrite (2) into the following equation:

$$Z_{t,x} = \beta_1 t + \beta_2 W_{t,x} + \int_{\mathbb{R}} zN(t,x,dz). \tag{7}$$

Let $\beta_1 = 0$; then (1) can be written as

$$\begin{aligned} du(t,x) = {} & [\lambda \Delta_\alpha u(t,x) + f(t,x,u_t)]\,dt \\ & + h(t,x,u_t)\,dW_{t,x} \\ & + g(t,x,u_t)\,dY_{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & \quad u(\eta,x) = \phi(\eta,x), \quad \eta \in (r,0], \end{aligned} \tag{8}$$

where $W_{t,x}$ is a one-dimensional space-time white noise and $Y_{t,x} := \int_{\mathbb{R}} zN(t,x,dz)$ is a one-dimensional pure jump Lévy process with Lévy measure of ν. We suppose that W generates a $\{\mathcal{F}\}_{t\geq 0}$-martingale measure in the sense of Walsh [15].

The following assumptions are imposed to the initial data u_0 and $\phi(\eta,x)$, $f(u_t)$, $h(u_t)$, and $g(u_t)$ to show the existence and uniqueness of the mild solution.

(H1) The initial data u_0 which is $\mathcal{F}_0$-measurable and ϕ satisfy

$$\sup_{x\in\mathbb{R}} |u(0,x)|^2 < \infty, \qquad \sup_{x\in\mathbb{R}} \int_{-r}^{0} |\phi(\eta,x)|^2 d\eta < \infty. \tag{9}$$

(H2) There exists a constant K such that for all $t \geq 0$,

$$\begin{aligned} & |f(u_t)|^2 + |h(u_t)|^2 + \left(\int_{\mathbb{R}} |g(u_t)|\,|z|\,\nu(dz)\right)^2 \\ & \quad < K\left(|u|^2 + \int_{-r}^{0} |u(t+\eta)|^2 d\eta\right), \\ & |f(u_t) - f(v_t)|^2 + |h(u_t) - h(v_t)|^2 \\ & \quad + \left(\int_{\mathbb{R}} |g(u_t) - g(v_t)|\,|z|\,\nu(dz)\right)^2 \\ & \quad < K\left(|u-v|^2 + \int_{-r}^{0} |u(t+\eta) - v(t+\eta)|^2 d\eta\right). \end{aligned} \tag{10}$$

Let the Green kennel $G_\alpha(t,x)$ be the fundamental solution of the Cauchy problem:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \lambda \Delta_\alpha v, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ v(0,x) &= \delta_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{11}$$

where $\delta_0(x)$ denotes the Dirac function. By Fourier transform,

$$G_\alpha(t,x) = \left[\mathcal{F}^{-1}\left(e^{\lambda t|\cdot|^\alpha}\right)\right](x). \tag{12}$$

A higher order fractional Green kennel is introduced in [16].

The following lemma gives some useful properties about $G_\alpha(t,x)$, which are key technique tools to get the estimation for the existence and uniqueness of the mild solution.

Lemma 1 (see [7]). *The Green kernel function $G_\alpha(t,x)$ satisfies the following properties.*

(1) *For any $t \geq 0$ $G_\alpha(t,x) = t^{-1/\alpha} G_\alpha(1, t^{-1/\alpha}x)$.*

(2) *For $n \in (1/(\alpha+1), \alpha+1)$, $\int_0^T \int_{\mathbb{R}} |G_\alpha(t,x)|^n dt\, dx < \infty$.*

(3) *For any $x \in \mathbb{R}$, $\int_{\mathbb{R}} G_\alpha(t,x) dx = 1$.*

(4) *For any $t, s \in \mathbb{R}$, $G_\alpha(t,x) * G_\alpha(s,x) = G_\alpha(t+s,x)$.*

(5) *For any $x \in \mathbb{R}$, there exists a constant C such that*

$$\begin{aligned} G_\alpha(1,x) &\leq \frac{C}{1+|x|^{1+\alpha}}, \\ \partial_x^m G_\alpha(1,x) &\leq C \frac{|x|^{\alpha+m-1}}{1+|x|^{1+\alpha}}. \end{aligned} \tag{13}$$

3. Existence of the Mild Solution

In this section, we will first construct the proper working function space.

Let T be a fixed positive time and $\mathbb{B}$ the class of all $\mathscr{F}_t$-adapted càdlàg process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\left[|u(t,x)|^2\right] < \infty. \tag{14}$$

Let $\lambda > 0$ be arbitrarily fixed; we define

$$\begin{aligned} |u|_\lambda^2 = &\left\{ \int_0^T e^{-\lambda t} \sup_{x\in\mathbb{R}} E|u(t,x)|^2 dt \right\}_{t\geq 0} \\ &+ \left\{ \int_{-r}^0 e^{-\lambda t} E|u(t,x)|^2 dt \right\}_{t\in(-r,0)}. \end{aligned} \tag{15}$$

For any $u \in \mathbb{B}$, $|u|_\lambda^2 < \infty$. It is easy to verify that $|\cdot|_\lambda$ is a norm and $(\mathbb{B}, |\cdot|_\lambda)$ is a Banach space.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ and $G_\alpha(t,x)$ be given as in the previous section. Following the idea in [17], we represent a mild solution of (8) for $t \geq 0$.

Definition 2. An $\mathscr{F}_t$-adapted random field $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is said to be a mild solution of (8) with initial value u_0 satisfying (H1) if the following integral equation is fulfilled:

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} G_\alpha(t,x-y) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) f(u_s) dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) h(u_s) W(dy,ds) \\ &+ \int_0^t \iint_{\mathbb{R}} G_\alpha(t-s,x-y) g(u_s) zN(ds,dy,dz), \end{aligned} \tag{16}$$

where the stochastic integral with respect to $W(t,x)$ is understood in the sense of that introduced by Walsh [15].

Theorem 3. *For $t \geq 0$ and $\alpha \in (1,2]$, assume that (H1) and (H2) hold, then there exists a unique mild solution $u \in \mathbb{B}$ for* (8).

Remark 4. In the following proof, C is a local constant which may change from line to line.

Proof. We will prove the theorem by the following two steps.

Step 1. Suppose that $u \in \mathbb{B}$ and denote

$$\begin{aligned} \mathscr{T}u(t,x) = &\int_{\mathbb{R}} G_\alpha(t,x-y) u_0(y) dy + \mathscr{T}_1 u(t,x) \\ &+ \mathscr{T}_2 u(t,x) + \mathscr{T}_3 u(t,x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mathscr{T}_1 u(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) f(u_s) dy\, ds, \\ \mathscr{T}_2 u(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) h(u_s) W(dy,ds), \\ \mathscr{T}_3 u(t,x) &= \int_0^t \iint_{\mathbb{R}} G_\alpha(t-s,x-y) \\ &\quad \times g(u_s) zN(ds,dy,dz). \end{aligned} \tag{18}$$

It follows from Hölder's inequality, Lemma 1, (H1), and (H2) that

$$\begin{aligned} &E\left|\mathscr{T}_1 u(t,x)\right|^2 \\ &= E\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) f(u_s) dy\, ds\right|^2 \\ &\leq C \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) dy\, ds \\ &\quad \times \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) E\left|f(u_s)\right|^2 dy\, ds \\ &\leq CK \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) \\ &\quad \times E\Bigg(\left|u(s,y)\right|^2 \\ &\quad + \int_{-r}^0 \left|u(s+\eta,y)\right|^2 d\eta \Bigg) dy\, ds \\ &\leq C \int_0^t \sup_{y\in\mathbb{R}} E\Bigg(\left|u(s,y)\right|^2 \\ &\quad + \int_{-r}^0 \left|u(s+\eta,y)\right|^2 d\eta \Bigg) ds \\ &\leq Ct + C \int_0^t \int_{-r}^0 \sup_{y\in\mathbb{R}} E\left|u(s+\eta,y)\right|^2 d\eta\, ds \\ &\leq Ct + C \int_{-r}^0 \int_{-r}^t \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds\, d\eta \end{aligned}$$

$$
\begin{aligned}
&\leq Ct + Cr\int_{-r}^{0}\sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \\
&\quad + Cr\int_{0}^{t}\sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \\
&\leq Ct(r+1) + Cr < \infty.
\end{aligned}
\tag{19}
$$

Applying Burkholder-Davis-Gundy inequality, Lemma 1, (H1), and (H2), we have

$$
\begin{aligned}
&E\left|\mathcal{T}_2 u(t,x)\right|^2 \\
&\quad = E\left|\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\,h(u_s)\,W(ds\,dy)\right|^2 \\
&\quad \leq C\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)\,E\left|h(u_s)\right|^2 ds\,dy \\
&\quad \leq CK\int_0^t (t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}} E\left(\left|u(s,y)\right|^2 + \int_{-r}^{0}\left|u(s+\eta,y)\right|^2 d\eta\right) ds \\
&\quad \leq Ct^{1-(1/\alpha)} + C\int_{-r}^{0}\int_{-r}^{t}\left|(t+\eta-s)\right|^{-1/\alpha} \times \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds\,d\eta \\
&\quad = Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{t+\eta}\int_{-r}^{0}(t+\eta-s)^{-1/\alpha}d\eta \times \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \right.\\
&\qquad \left. + \int_{t+\eta}^{t}\int_{-r}^{0}(s-\eta-t)^{-1/\alpha}d\eta \times\sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds\right) \\
&\quad \leq Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{t+\eta}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \right.\\
&\qquad \left. + \int_{t+\eta}^{t}(s+r-t)^{1-(1/\alpha)} \times\sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds\right) \\
&\quad = Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{0}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \right.\\
&\qquad + \int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds \\
&\qquad \left. + \int_{t+\eta}^{t}(s+r-t)^{1-(1/\alpha)} \times\sup_{y\in\mathbb{R}} E\left|u(s,y)\right|^2 ds\right) \\
&\quad \leq C\left(t^{1-(1/\alpha)} + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} + r^{2-(1/\alpha)}\right) < \infty,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&E\left|\mathcal{T}_3 u(t,x)\right|^2 \\
&\quad = E\left|\int_0^t\iint_{\mathbb{R}} G_\alpha(t-s,x-y)\,g(u_s)\,zN(ds,dy,dz)\right|^2 \\
&\quad \leq C\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y) \times E\left(\int_{\mathbb{R}}\left|g(u_s)\right|\left|z\right|\nu(dz)\right)^2 dy\,ds \\
&\quad \leq CK\int_0^t (t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}} E\left(\left|u(s,y)\right|^2 + \int_{-r}^{0}\left|u(s+\eta,y)\right|^2 d\eta\right) ds \\
&\quad \leq C\left(t^{1-(1/\alpha)} + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} + r^{2-(1/\alpha)}\right) < \infty.
\end{aligned}
\tag{21}
$$

Thus, combining (19) and (20) with (21), we derive

$$
\begin{aligned}
E\left|u(t,x)\right|^2 \leq C&\left[t(r+1) + r + t^{1-(1/\alpha)}\right.\\
&+ (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} \\
&\left. + r^{2-(1/\alpha)}\right].
\end{aligned}
\tag{22}
$$

Taking Laplace transform formula and (22), we deduce that

$$
\begin{aligned}
&\left|\mathcal{T}u(t,x)\right|_\lambda^2 \\
&\quad = \int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}}\left|\mathcal{T}u(t,x)\right|^2 dt \\
&\quad \leq C\int_0^\infty e^{-\lambda t}\left[t(r+1) + r + t^{1-(1/\alpha)} + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} + r^{2-(1/\alpha)}\right] dt \\
&\quad \leq C\left[(r+1)\Gamma(2)\lambda^{-2} + r\lambda^{-1} + \left(e^r+1\right)\Gamma\left(2-\frac{1}{\alpha}\right)\lambda^{-(2-(1/\alpha))}\right.
\end{aligned}
$$

$$+\Gamma\left(3-\frac{1}{\alpha}\right)\lambda^{-(3-(1/\alpha))}$$
$$+r^{2-(1/\alpha)}\lambda^{-1}\Big]$$
$$\leq C\left[\lambda^{-2}+\lambda^{-1}+\lambda^{-(2-(1/\alpha))}\right.$$
$$\left.+\lambda^{-(3-(1/\alpha))}\right]<\infty; \tag{23}$$

that is, $\mathcal{T}u \in \mathbb{B}$, which implies that operator $\mathcal{T}:\mathbb{B}\to\mathbb{B}$.

Step 2. For any $u, v \in \mathbb{B}$ and $t \geq 0$, it follows from Hölder's inequality, Lemma 1, and (H2) that

$$E\left|\mathcal{T}_1u(t,x)-\mathcal{T}_1v(t,x)\right|^2$$
$$=E\left|\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\left(f(u_s)-f(v_s)\right)dy\,ds\right|^2$$
$$\leq C\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\,dy\,ds$$
$$\times\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\,E\left|f(u_s)-f(v_s)\right|^2dy\,ds$$
$$\leq CK\int_0^t\sup_{y\in\mathbb{R}}E\left(\left|u(s,y)-v(s,y)\right|^2\right.$$
$$\left.+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\right)ds$$
$$\leq C\left(\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$
$$\left.+\int_{-r}^0\int_{-r}^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\,d\eta\right)$$
$$=C\left((r+1)\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$
$$\left.+r\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right). \tag{24}$$

By Burkholder-Davis-Gundy inequality, Lemma 1 and (H2), we have

$$E\left|\mathcal{T}_2u(t,x)-\mathcal{T}_2v(t,x)\right|^2$$
$$=E\left|\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\left(h(u_s)-h(v_s)\right)W(dy\,ds)\right|^2$$
$$\leq C\int_0^t\int_{\mathbb{R}}G_\alpha^2(t-s,x-y)\,E\left|h(u_s)-h(v_s)\right|^2dy\,ds$$
$$\leq CK\int_0^t(t-s)^{-(1/\alpha)}$$
$$\times\sup_{y\in\mathbb{R}}E\left(\left|u(s,y)-v(s,y)\right|^2\right.$$
$$\left.+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\right)ds$$
$$\leq C\left(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$
$$+\int_{-r}^0\int_{-r}^t\left|t+\eta-s\right|^{-1/\alpha}$$
$$\left.\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\,d\eta\right)$$
$$\leq C\left(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$
$$+\int_{-r}^{t+\eta}\int_{-r}^0(t+\eta-s)^{-1/\alpha}$$
$$\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2d\eta\,ds$$
$$+\int_{t+\eta}^t\int_{-r}^0(s-\eta-t)^{-1/\alpha}$$
$$\left.\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2d\eta\,ds\right)$$
$$\leq C\left(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$
$$+\int_{-r}^0(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds$$
$$+\int_0^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds$$
$$\left.+r^{1-(1/\alpha)}\int_{t+\eta}^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right),$$

$$E\left|\mathcal{T}_3u(t,x)-\mathcal{T}_3v(t,x)\right|^2$$
$$=E\left|\int_0^t\iint_{\mathbb{R}}G_\alpha(t-s,x-y)\right.$$
$$\left.\times\left(g(u_s)-g(v_s)\right)zN(ds,dy,dz)\right|^2$$
$$\leq C\int_0^t\int_{\mathbb{R}}G_\alpha^2(t-s,x-y)$$
$$\times E\left(\int_{\mathbb{R}}\left|g(u_s)-g(v_s)\right||z|\,\nu(dz)\right)^2dy\,ds$$
$$\leq CK\int_0^t(t-s)^{-1/\alpha}$$
$$\times\sup_{y\in\mathbb{R}}E\left(\left|u(s,y)-v(s,y)\right|^2\right.$$
$$\left.+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\right)ds$$
$$\leq C\left(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\right.$$

$$+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds$$
$$+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds$$
$$\left.+r^{1-(1/\alpha)}\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\right). \tag{25}$$

Thus, it follows that

$$\begin{aligned}
&E\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|^{2}\\
&\leq C\Bigg((r+1)\int_{0}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\quad+r\int_{-r}^{0}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\quad+\int_{0}^{t}(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\quad+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\quad+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\quad+r^{1-(1/\alpha)}\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg).
\end{aligned} \tag{26}$$

Finally, direct computation implies that

$$\begin{aligned}
&\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|_{\lambda}^{2}\\
&=\int_{0}^{T}e^{-\lambda t}\sup_{y\in\mathbb{R}}E\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|^{2}\\
&\leq C\int_{0}^{T}e^{-\lambda t}\Bigg((r+1)\\
&\qquad\times\int_{0}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r\int_{-r}^{0}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t}(t-s)^{-1/\alpha}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r^{1-(1/\alpha)}\\
&\qquad\times\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg)dt\\
&\leq C\Bigg[\int_{0}^{\infty}e^{-\lambda t}dt\\
&\qquad\times\Bigg((r+1)\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad\quad+r\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg)\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{-1/\alpha}dt\\
&\qquad\times\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{1-(1/\alpha)}dt\\
&\qquad\times\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{1-(1/\alpha)}dt\\
&\qquad\times\int_{0}^{T+\eta}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}dt\times r^{1-(1/\alpha)}\\
&\qquad\times\int_{T+\eta}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg]\\
&\leq C\Bigg[\left(\frac{r+1}{\lambda}+\frac{\Gamma(1-(1/\alpha))}{\lambda^{1-(1/\alpha)}}\right)\\
&\qquad\times\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\left(\frac{r}{\lambda}+\frac{\Gamma(2-(1/\alpha))}{\lambda^{2-(1/\alpha)}}\right)\\
&\qquad\times\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\frac{\Gamma(2-(1/\alpha))}{\lambda^{2-(1/\alpha)}}\\
&\qquad\times\int_{0}^{T+\eta}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds
\end{aligned}$$

$$+\frac{r^{1-(1/\alpha)}}{\lambda}\int_{T+\eta}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg]$$

$$\le\frac{C_1}{\lambda^{K_1}}\left(\int_0^T e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2 ds\right.$$

$$\left.+\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2 ds\right)$$

$$=\frac{C_1}{\lambda^{K_1}}|u-v|_\lambda^2, \tag{27}$$

where $C_1 = 2C\cdot\max\{r+1,\Gamma(1-(1/\alpha)),\Gamma(2-(1/\alpha)),r^{1-(1/\alpha)}\}$ and $K_1 = 1-(1/\alpha)$.

Let λ that large enough such that

$$\frac{C_1}{\lambda^{K_1}}<1, \tag{28}$$

which implies that the operator $\mathcal{T}:\mathbb{B}\to\mathbb{B}$ is contraction. By the Banach fixed point theorem, there exists a unique fixed point in $\mathbb{B}$. Moreover, the fixed point is the unique mild solution of (8). □

Remark 5. If there are no delays, Theorem 3 can be solved in the following working function space:

$$|u|_\lambda^2=\int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}}E|u(t,x)|^2dt, \tag{29}$$

where $u\in\mathbb{B}$, which implies that the delays affect the working function space.

4. The Regularity of the Mild Solution

In this section, we will show the time regularity and space regularity of the mild solution for (8). In order to prove the regularity, we need the following assumptions:

(H3) there exists some $\gamma<1/2$

$$\sup_{x\in\mathbb{R}}E\left(\left|u_0(x+z)-u_0(x)\right|^2\right)<c|z|^{2\gamma}; \tag{30}$$

(H4) for $\theta>0$, let $|\phi(\eta,x)|<\infty$ and there exists some $\pi<1$ such that

$$\sup_{x\in\mathbb{R}}\int_{-r}^{-\theta}E\left|u(\eta+\theta,x)-u(\eta,x)\right|^2d\eta<c|\theta|^{2\pi}; \tag{31}$$

(H5) $|f(t_1,y,u_{t_1})-f(t_2,y,u_{t_2})|^2\le c(|t_1-t_2|^2+|u(t_1)-u(t_2)|^2+\int_{-r}^{0}|u(t_1+\eta)-u(t_2+\eta)|^2d\eta)$.

To the end, we will give an important lemma from [7].

Lemma 6. (1) *For* $1<n<\alpha+1$, $\int_0^\infty\int_{\mathbb{R}}|G_\alpha(1+v,z)-G_\alpha(v,z)|^n dz\,dv<\infty$.

(2) *For* $(\alpha+1)/2<n<\alpha+1$, $\int_0^\infty\int_{\mathbb{R}}|G_\alpha(v,z+1)-G_\alpha(v,z)|^n dz\,dv<\infty$.

Theorem 7. *Assume that the conditions (H1)–(H5) are satisfied; then for* $\alpha\in(1,2]$ *and* $t\ge0$, *there exists a continuous modification* $u(t,x)$, *which is* β-*Hölder continuous in* t, *where* $\beta=\min\{\gamma/\alpha,\pi,(1/2)-(1/2\alpha)\}$.

Proof. For $t\ge0$, it follows that, for any $x\in\mathbb{R}$ and $\theta>0$,

$$\begin{aligned}
&|u(t+\theta,x)-u(t,x)|\\
&\le\left|\int_{\mathbb{R}}\left(G_\alpha(t+\theta,x-y)-G_\alpha(t,x-y)\right)u_0(y)\,dy\right|\\
&\quad+\left|\int_0^{t+\theta}\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)\right.\right.\\
&\qquad\left.\left.-G_\alpha(t-s,x-y)\right)f(u_s)\,dy\,ds\right|\\
&\quad+\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)\right.\right.\\
&\qquad\left.\left.-G_\alpha(t-s,x-y)\right)h(u_s)\,W(ds\,dy)\right|\\
&\quad+\left|\int_t^{t+\theta}\int_{\mathbb{R}}G_\alpha(t+\theta-s,x-y)\,h(u_s)\,W(ds\,dy)\right|\\
&\quad+\left|\int_0^t\iint_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)-G_\alpha(t-s,x-y)\right)\right.\\
&\qquad\left.\times g(u_s)\,zN(ds,dz)\,dy\right|\\
&\quad+\left|\int_t^{t+\theta}\iint_{\mathbb{R}}G_\alpha(t+\theta-s,x-y)\,g(u_s)\,zN(ds,dz)\,dy\right|\\
&=\phi_\theta^0+\phi_\theta^1+\phi_\theta^2+\phi_\theta^3+\phi_\theta^4+\phi_\theta^5.
\end{aligned} \tag{32}$$

Next, we will estimate each term ϕ_θ^j $(j=0,1,\ldots,5)$, respectively.

Combining Hölder inequality, Lemma 1 with (H3) yields

$$\begin{aligned}
&E\left|\phi_\theta^0\right|^2\\
&=E\left|\int_{\mathbb{R}}G_\alpha(\theta,z)\left[\int_{\mathbb{R}}G_\alpha(t,x-y)\right.\right.\\
&\qquad\left.\left.\times\left(u_0(y-z)-u_0(y)\right)dy\right]dz\right|^2\\
&\le CE\left(\int_{\mathbb{R}}G_\alpha(\theta,z)\left[\int_{\mathbb{R}}G_\alpha(t,x-y)\right.\right.\\
&\qquad\left.\left.\times\left(u_0(y-z)-u_0(y)\right)dy\right]^2dz\right)\\
&\quad\times\left(\int_{\mathbb{R}}G_\alpha(\theta,z)\,dz\right)
\end{aligned}$$

$$\leq C\int_{\mathbb{R}} G_\alpha(\theta,z)\sup_{y\in\mathbb{R}} E\left|u_0(y-z)-u_0(y)\right|^2 dz$$
$$\leq C\int_{\mathbb{R}} G_\alpha(\theta,z)|z|^{2\gamma}dz = C\int_{\mathbb{R}}\theta^{2\gamma/\alpha}G_\alpha(1,z)z^{2\gamma}dz$$
$$\leq C\theta^{2\gamma/\alpha}\int_{\mathbb{R}}\frac{|z|^{2\gamma}}{1+|z|^{1+\alpha}}dz \leq C\theta^{2\gamma/\alpha}. \tag{33}$$

Next, we consider ϕ_θ^1. Let $s'=s-\theta$; then,

$$\begin{aligned}\left|\phi_\theta^1\right| \leq{}& \left|\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\right.\\ &\left.\times\left(f(s+\theta,y,u_{s+\theta})-f(s,y,u_s)\right)dy\,ds\right|\\ &+\left|\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\right.\\ &\left.\times f(s,y,u_s)\,dy\,ds\right|\\ ={}& \phi_\theta^{1.1}+\phi_\theta^{1.2}.\end{aligned} \tag{34}$$

By Hölder inequality, Lemma 1, (H4), and (H5), we have

$$\begin{aligned}&E\left|\phi_\theta^{1.1}\right|^2\\ &\leq C\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\,dy\,ds\\ &\quad\times\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\\ &\quad\times E\left|f(s+\theta,y,u_{s+\theta})-f(s,y,u_s)\right|^2 dy\,ds\\ &\leq C\int_0^t\left(\theta^2+\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2\right.\\ &\quad+\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s+\theta+\eta,y)\right.\\ &\quad\left.\left.-u(s+\eta,y)\right|^2 d\eta\right)ds\\ &\leq C\left(T\theta^2+(r+1)\right.\\ &\quad\times\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2 ds\\ &\quad+r\int_{-r}^{-\theta}\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2 ds\\ &\quad\left.+\int_{-\theta}^0\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2 ds\right)\\ &\leq C\left(\theta^2+\theta^{2\pi}+\theta\right.\\ &\quad\left.+\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2 ds\right),\end{aligned}$$

$$\begin{aligned}&E\left|\phi_\theta^{1.2}\right|^2\\ &\leq C\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,dy\,ds\\ &\quad\times\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)E\left|f(u_s)\right|^2 dy\,ds\\ &\leq C\theta\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)E\left|f(u_s)\right|^2 dy\,ds\\ &\leq C\theta\int_0^\theta\sup_{y\in\mathbb{R}}E\left(\left|u(s,y)\right|^2+\int_{-r}^0\left|u(s+\eta,y)\right|^2 d\eta\right)ds\\ &\leq C\theta\left((r+1)\int_0^\theta\sup_{y\in\mathbb{R}}E\left|u(s,y)\right|^2 ds\right.\\ &\quad\left.+r\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s,y)\right|^2 ds\right)\\ &\leq C\left(\theta^2+\theta\right).\end{aligned} \tag{35}$$

Taking the transformation $s=\theta v$, $y=\theta^{1/\alpha}z$, by Burkholder-Davis-Gundy inequality, Lemmas 1 and 6, and (H2), we obtain

$$\begin{aligned}&E\left|\phi_\theta^2\right|^2\\ &\leq C\int_0^t\int_{\mathbb{R}} E\left(G_\alpha(t+\theta-s,x-y)\right.\\ &\quad\left.-G_\alpha(t-s,x-y)\right)^2 h^2(u_s)\,ds\,dy\\ &\leq CK\int_0^t\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)\right.\\ &\quad\left.-G_\alpha(t-s,x-y)\right)^2\\ &\quad\times E\left(|u(s)|^2+\int_{-r}^0\left|u(s+\eta)\right|^2 d\eta\right)dy\,ds\\ &\leq C\int_0^t\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)-G_\alpha(t-s,x-y)\right)^2\\ &\quad\times E\left(|u(s)|^2+\int_{-r}^0|u(\eta)|^2 d\eta\right.\\ &\quad\left.+\int_0^t\left|u(\eta')\right|^2 d\eta'\right)dy\,ds\\ &\leq C\left((T+1)\sup_{[0,T]\times\mathbb{R}}E\left|u(s,y)\right|^2+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u(\eta,y)\right|^2 d\eta\right)\\ &\quad\times\int_0^t\int_{\mathbb{R}}\left(G_\alpha(s+\theta,y)-G_\alpha(s,y)\right)^2 dy\,ds\\ &\leq C\left(\theta^{-1/\alpha}\theta\int_0^\infty\int_{\mathbb{R}}\left(G_\alpha(v+1,z)-G_\alpha(v,z)\right)^2 dz\,dv\right)\\ &\leq C\theta^{1-(1/\alpha)},\end{aligned}$$

$$
\begin{aligned}
&E\left|\phi_\theta^3\right|^2\\
&\quad\leq C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right.\\
&\qquad\qquad\left.-G_\alpha\left(t-s,x-y\right)\right)^2\left|h\left(u_s\right)\right|^2 ds\,dy\\
&\quad\leq C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right.\\
&\qquad\qquad\left.-G_\alpha\left(t-s,x-y\right)\right)^2\\
&\qquad\qquad\times\left(\left|u\left(s,y\right)\right|^2+\int_{-r}^0\left|u\left(s+\eta\right)\right|^2 d\eta\right) dy\,ds\\
&\quad\leq C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right.\\
&\qquad\qquad\left.-G_\alpha\left(t-s,x-y\right)\right)^2\\
&\qquad\qquad\times\left(\left|u\left(s,y\right)\right|^2+\int_{-r}^{t+\theta}\left|u\left(\eta'\right)\right|^2 d\eta'\right) dy\,ds\\
&\quad\leq C\left(\left(T+1+\theta\right)\sup_{[0,T+\theta]\times\mathbb{R}} E\left|u\left(s,y\right)\right|^2\right.\\
&\qquad\qquad\left.+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u\left(\eta,y\right)\right|^2 d\eta\right)\\
&\qquad\qquad\times\int_0^\theta\int_{\mathbb{R}} E\left(G_\alpha\left(s+\theta,y\right)-G_\alpha\left(s,y\right)\right)^2 dy\,ds\\
&\quad\leq C\left(1+\theta\right)\theta^{1-(1/\alpha)}\\
&\qquad\qquad\times\int_0^1\int_{\mathbb{R}} E\left(G_\alpha\left(1+v,z\right)-G_\alpha\left(v,z\right)\right)^2 dz\,dv\\
&\quad\leq C\left(\theta^{1-(1/\alpha)}+\theta^{2-(1/\alpha)}\right).
\end{aligned}
\tag{36}
$$

Then, by the same method, we have

$$
\begin{aligned}
E\left|\phi_\theta^4\right|^2 \leq{}& \int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t+\theta-s,x-y\right)-G_\alpha\left(t-s,x-y\right)\right)^2\\
&\qquad\times E\left(\int_{\mathbb{R}}\left|g\left(u_s\right)\right|\left|z\right|\nu\left(dz\right)\right)^2 ds\,dy\\
&\leq CK\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t+\theta-s,x-y\right)\right.\\
&\qquad\left.-G_\alpha\left(t-s,x-y\right)\right)^2\\
&\qquad\times E\left(\left|u\left(s\right)\right|^2+\int_{-r}^0\left|u\left(s+\eta\right)\right|^2 d\eta\right) dy\,ds\\
&\leq C\theta^{1-(1/\alpha)},\\
&\qquad E\left|\phi_\theta^5\right|^2\leq C\left(\theta^{1-(1/\alpha)}+\theta^{2-(1/\alpha)}\right).
\end{aligned}
\tag{37}
$$

Thus, from the previous estimates, let $\beta=\min\{\gamma/\alpha,\pi,(1/2)-(1/2\alpha)\}$:

$$
\begin{aligned}
&E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2\\
&\quad\leq C\left[\theta^{2\beta}+\int_0^t\sup_{y\in\mathbb{R}}E\left|u\left(s+\theta,y\right)-u\left(s,y\right)\right|^2 ds\right].
\end{aligned}
\tag{38}
$$

Hence, it follows from Gronwall's Lemma that

$$
E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2\leq C\theta^{2\beta}.
\tag{39}
$$

Then, for $t\geq 0$, we have

$$
\begin{aligned}
&\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|_\lambda^p\\
&\quad=\left(\int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}}E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2 ds\right)^{p/2}\\
&\quad\leq C\theta^{\beta p}.
\end{aligned}
\tag{40}
$$

□

Finally, we study the space regularity of the mild solution for (8).

Theorem 8. *Assume that the conditions (H1)–(H3) are satisfied; then for $\alpha\in(1,2]$ and $t\geq 0$, there exists a continuous modification $u(t,x)$, which is ρ-Hölder continuous in x, where $\rho=\min\{\gamma,\vartheta,\alpha-1\}$.*

Proof. It follows that, for any $t\in[0,T]$ and $\zeta>0$,

$$
\begin{aligned}
&\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right|\\
&\quad\leq\left|\int_{\mathbb{R}}\left(G_\alpha\left(t,x+\zeta-y\right)-G_\alpha\left(t,x-y\right)\right)u_0\left(y\right)dy\right|\\
&\qquad+\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right.\\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)f\left(u_s\right)dy ds\right|\\
&\qquad+\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right.\\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)h\left(u_s\right)W\left(ds dy\right)\right|\\
&\qquad+\left|\int_0^t\iint_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right.\\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)g\left(u_s\right)zN\left(ds,dz\right)dy\right|\\
&\quad:=\sum_{j=0}^3\phi_\zeta^j.
\end{aligned}
\tag{41}
$$

By (H3) and Lemma 1, we have

$$E\left|\phi_\zeta^0\right|^2 = E\left|\int_{\mathbb{R}} G_\alpha(t, x-y)\left[u_0(y+\zeta) - u_0(y)\right] dy\right|^2 \leq \sup_{y\in\mathbb{R}} E\left(\left|u_0(y+\zeta) - u_0(y)\right|^2\right) \times \left(\int_{\mathbb{R}} G_\alpha(t, x-y)\, dy\right)^2 \leq C\zeta^{2\gamma}. \tag{42}$$

By (H2), Hölder's inequality and Lemma 1, we set $\epsilon = (1/2)(\alpha+1) - \delta$ (δ is small enough) and $\vartheta \in (0,1)$ and we can derive

$$\begin{aligned}
E\left|\phi_\zeta^1\right|^2 &\leq C\left((1+T)\sup_{[0,T]\times\mathbb{R}} E|u(t,x)|^2 + \sup_{y\in\mathbb{R}}\int_{-r}^0 |u(\eta,y)|^2 d\eta\right) \\
&\quad \times \left|\int_0^t \int_{\mathbb{R}} \left(G_\alpha(s, y+\zeta) - G_\alpha(s,y)\right) dy\, ds\right|^2 \\
&\leq C\left|\int_0^t \int_{\mathbb{R}} s^{-1/\alpha}\left(G_\alpha\left(1, s^{-1/\alpha}(y+\zeta)\right) - G_\alpha(1, s^{-1/\alpha}y)\right) dy\, ds\right|^2 \\
&= C\left|\int_0^t \int_{\mathbb{R}} s^{-\epsilon/\alpha}\left(G_\alpha\left(1, s^{-1/\alpha}(y+\zeta)\right) - G_\alpha\left(1, s^{-1/\alpha}y\right)\right)^{(1-\vartheta)} \right. \\
&\quad \left. \times s^{-((1-\epsilon)/\alpha)}\left(G_\alpha\left(1, s^{-1/\alpha}(y+\zeta)\right) - G_\alpha\left(1, s^{-1/\alpha}y\right)\right)^{\vartheta} dy\, ds\right|^2 \\
&\leq C\left(\int_0^t \int_{\mathbb{R}} s^{-2\epsilon/\alpha}\left|G_\alpha\left(1, s^{-1/\alpha}(y+\zeta)\right) - G_\alpha\left(1, s^{-1/\alpha}y\right)\right|^{2(1-\vartheta)} dy\, ds\right) \\
&\quad \times \left(\int_0^t \int_{\mathbb{R}} s^{-2(1-\epsilon)/\alpha}\left|G_\alpha\left(1, s^{-1/\alpha}(y+\zeta)\right) - G_\alpha\left(1, s^{-1/\alpha}y\right)\right|^{2\vartheta} dy\, ds\right) \\
&:= cI \times II.
\end{aligned} \tag{43}$$

By Lemma 1, recall that $2\epsilon < \alpha + 1$, and we have

$$I \leq C\left(\int_0^t \int_{\mathbb{R}} s^{-(2\epsilon-1)/\alpha}\left(G_\alpha(1,y)\right)^{2(1-\vartheta)} dy\, ds\right) \leq C\int_0^t s^{-(2\epsilon-1)/\alpha} ds < \infty. \tag{44}$$

Then, by the mean value of the theorem and Lemma 1, we have that

$$\begin{aligned}
II &= \int_0^t \int_{\mathbb{R}} s^{-2(1-\epsilon)/\alpha}\left(\frac{\partial}{\partial y} G_\alpha\left(1, s^{-1/\alpha}(y+\varepsilon)\right) \times s^{-1/\alpha}\zeta\right)^{2\vartheta} dy\, ds \quad (\varepsilon \in (0,\zeta)) \\
&= \zeta^{2\vartheta}\int_0^t \int_{\mathbb{R}} s^{-(2(1-\epsilon+\vartheta)-1)/\alpha} \times \left(\frac{\partial}{\partial y} G_\alpha(1,y)\right)^{2\vartheta} dy\, ds \\
&\leq C\zeta^{2\vartheta}\int_0^t s^{-(2(1-\epsilon+\vartheta)-1)/\alpha} ds.
\end{aligned} \tag{45}$$

Choosing $\vartheta < \alpha - \delta$ such that $(2(1-\epsilon+\vartheta)-1)/\alpha < 1$, then

$$II < C\zeta^{2\vartheta}. \tag{46}$$

Combining (44) and (46), we have

$$E\left|\phi_\zeta^1\right|^2 \leq C\zeta^{2\vartheta}. \tag{47}$$

Taking the change of variable $s = \zeta^\alpha v$, $y = z\zeta$, and by Burkholder-Davis-Gundy inequality, Lemmas 1 and 6, (H1), and (H2), we derive

$$\begin{aligned}
E\left|\phi_\zeta^2\right|^2 &\leq \left(\int_0^t \int_{\mathbb{R}} E\left(G_\alpha(t-s, x+\zeta-y) - G_\alpha(t-s, x-y)\right)^2 \times E\left|h(u_s)\right|^2 dy\, ds\right) \\
&\leq C\left((T+1)\sup_{[0,T]\times\mathbb{R}} E|u(t,x)|^2 + \sup_{y\in\mathbb{R}}\int_{-r}^0 |u(\eta,y)|^2 d\eta\right) \\
&\quad \times \left(\int_0^t \int_{\mathbb{R}} \left(G_\alpha(s, y+\zeta) - G_\alpha(s,y)\right)^2 dy ds\right) \\
&\leq C\left(\zeta^{-2}\zeta\zeta^\alpha \int_0^\infty \int_{\mathbb{R}} \left(G_\alpha(v, z+1) - G_\alpha(v,z)\right)^2 dz\, dv\right) \\
&\leq C\zeta^{2(\alpha-1)},
\end{aligned}$$

$$E\left|\phi_{\zeta}^{3}\right|^{2} \leq C\int_{0}^{t}\int_{\mathbb{R}}\left(G_{\alpha}\left(t-s,x+\zeta-y\right)-G_{\alpha}\left(t-s,x-y\right)\right)^{2}\times E\left(\int_{\mathbb{R}}\left|g\left(u_{s}\right)\right|z\nu\left(dz\right)\right)^{2}ds\,dy \leq C\zeta^{2(\alpha-1)}. \quad (48)$$

Combining (42)–(48), we have

$$E\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right|^{2} \leq C\left(\zeta^{2\gamma}+\zeta^{2\vartheta}+\zeta^{2(\alpha-1)}\right). \quad (49)$$

Then, we have, for $t \in [0,T]$,

$$\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right|_{\lambda}^{p} \leq C\left(\zeta^{\gamma p}+\zeta^{\vartheta p}+\zeta^{(\alpha-1)p}\right) \leq C\zeta^{\rho p}, \quad (50)$$

where $\rho = \min\{\gamma, \vartheta, \alpha - 1\}$. □

Remark 9. Theorems 7 and 8 show that the regularity of initial value and the order of fractional operator can affect both time regularity and space regularity of the mild solution for (1). In particular, the time regularity is affected by the regularity of initial value with delays.

Acknowledgments

This work is supported by the NSF of China (nos. 10971225 and 11371367) and NSF of Hunan Province (no. 11JJ3004).

References

[1] C. Bardos, P. Penel, U. Frisch, and P.-L. Sulem, "Modified dissipativity for a nonlinear evolution equation arising in turbulence," *Archive for Rational Mechanics and Analysis*, vol. 71, no. 3, pp. 237–256, 1979.

[2] L. Bo, K. Shi, and Y. Wang, "On a nonlocal stochastic Kuramoto-Sivashinsky equation with jumps," *Stochastics and Dynamics*, vol. 7, no. 4, pp. 439–457, 2007.

[3] B. Guo, X. Pu, and F. Huang, *Fractional Partial Differential Equations and Numerical Solution*, Science Press, Beijing, China, 2011.

[4] J. Huang, Y. Li, and Y. Zheng, *Introduction of Random Dynamical Systems*, Science Press, Beijing, China, 2012.

[5] J. M. Angulo, M. D. Ruiz-Medina, V. V. Anh, and W. Grecksch, "Fractional diffusion and fractional heat equation," *Advances in Applied Probability*, vol. 32, no. 4, pp. 1077–1099, 2000.

[6] P. Azerad and M. Mellouk, "On a stochastic partial differential equation with non-local diffusion," *Potential Analysis*, vol. 27, no. 2, pp. 183–197, 2007.

[7] L. Debbi and M. Dozzi, "On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension," *Stochastic Processes and Their Applications*, vol. 115, no. 11, pp. 1764–1781, 2005.

[8] A. Truman and J. Wu, *Fractal Burgers Equations Driven by Lévy Noise*, vol. 245 of *Lecture Notes in Pure and Applied Mathematics*, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2006.

[9] Z. Brzeźniak and L. Debbi, "On stochastic Burgers equation driven by a fractional Laplacian and space-time white noise," in *Stochastic Differential Equations: Theory and Applications*, vol. 2, pp. 135–167, World Scientific Publishing, Hackensack, NJ, USA, 2007.

[10] Z. Brzeźniak, L. Debbi, and B. Goldys, "Ergodic properties of fractional stochastic Burgers equation," Preprint, 2011, http://arxiv.org/pdf/1106.1918.pdf.

[11] T. Caraballo and J. Real, "Attractors for 2D-Navier-Stokes models with delays," *Journal of Differential Equations*, vol. 205, no. 2, pp. 271–297, 2004.

[12] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, vol. 119 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1996.

[13] M. Niu and B. Xie, "Regularity of a fractional partial differential equation driven by space-time white noise," *Proceedings of the American Mathematical Society*, vol. 138, no. 4, pp. 1479–1489, 2010.

[14] K. Kyeong-Hun and K. Panki, "An Lp-theory of stochastic parabolic equations with the random fractional Laplacian driven by Levy processes," 2011, http://arxiv.org/pdf/1111.4712.pdf.

[15] J. B. Walsh, "An introduction to stochastic partial differential equations," in *École d'été de Probabilités de Saint-Flour*, vol. 1180 of *Lecture Notes in Mathematics*, pp. 265–439, Springer, Berlin, Germany, 1986.

[16] L. Debbi, "On some properties of a high order fractional differential operator which is not in general selfadjoint," *Applied Mathematical Sciences*, vol. 1, no. 25-28, pp. 1325–1339, 2007.

[17] J. Walsh, *An Introduction to Stochastic Partial Differential Equations*, vol. 1180 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1986.